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# Developing mathematical fluency: exercises or rich tasks?

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# MATHEMATICAL FLUENCY *WITHOUT* DRILL AND PRACTICE

Colin Foster asks how can we avoid letting 'practice' dominate the teaching of the new mathematics national curriculum

## Introduction

The word 'practice' appears twice in the short 'Aims' section of the *KS3 Programme of study* (DfE, 2013). The first stated aim is that all pupils:

*… become **fluent** in the fundamentals of mathematics, including through varied and frequent practice with increasingly complex problems over time, so that pupils develop conceptual understanding and the ability to recall and apply knowledge rapidly and accurately.* (p. 2)

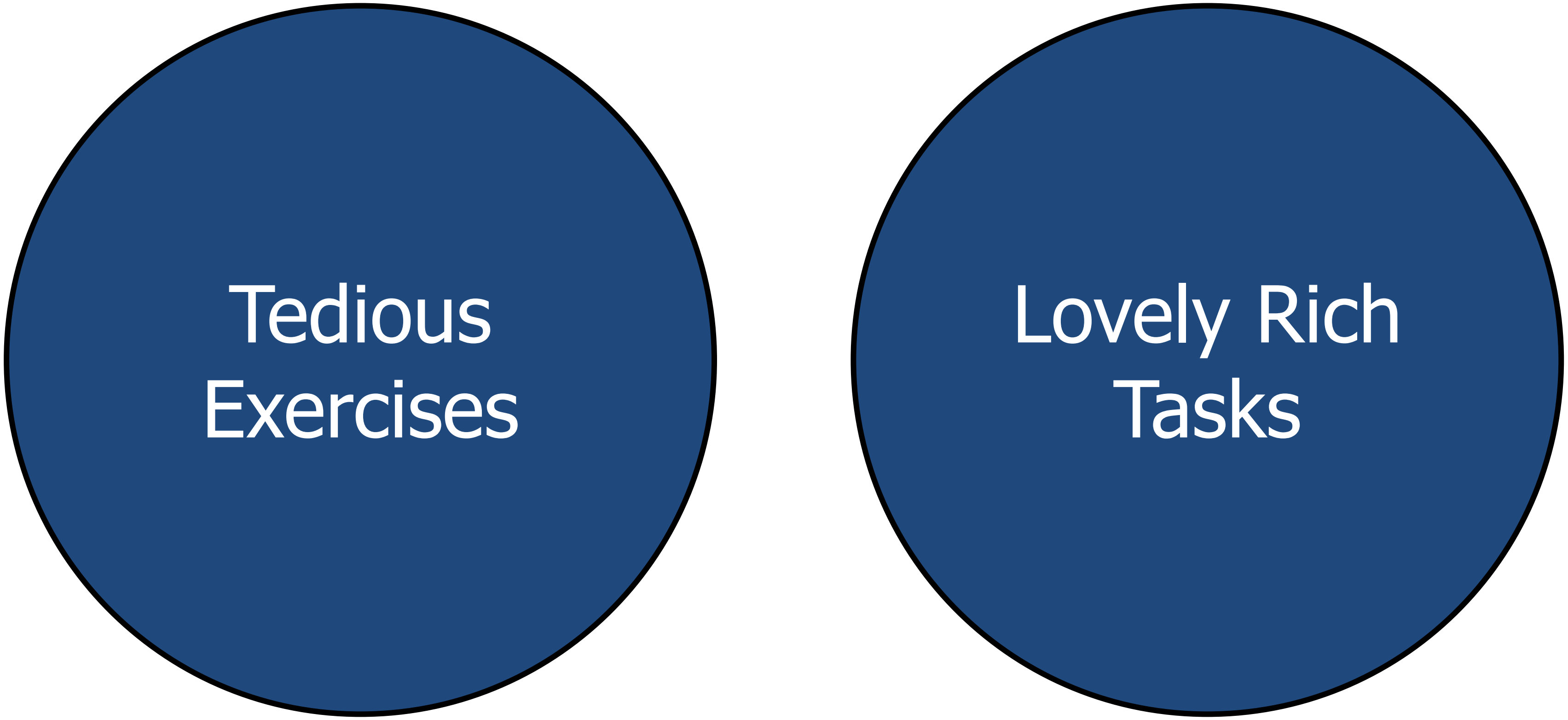
This optimistic sentence implies that focusing on fluency will lead eventually to conceptual understanding and confidence in applying the knowledge gained. This reminds me of John Holt's (1990) observation that:

*the notion that if a child repeats a meaningless*

I very much agree with. However, the following sentence, that '*Those who are not sufficiently fluent should consolidate their understanding, including through additional practice, before moving on*', sounds to me like a recipe for never-ending, low-level, imitative rehearsing of knowledge and skills until students earn the right to anything more stimulating.

It is easy to see how students can become trapped in tedious, repetitive work, endlessly 'practising the finished product' (Prestage and Perks, 2006). Teachers are going to be told that certain students 'need more practice on X' before they are 'ready' to move on. Students will be discouraged and demotivated by constant, unimaginative repetition and the low, or slow, achievement that has led to this judgment becomes a self-fulfilling prophecy.

# Mathematics Lessons



# Mathematics Lessons

**Tedious Exercises**

What are the factors of 16?

**Lovely Rich Tasks**

Find some numbers with exactly 5 factors.

# Procedural Fluency

Knowing when and how to apply a mathematical procedure and being able to perform it "accurately, efficiently, and flexibly" (NCTM, 2014, p. 1).

"The national curriculum for mathematics aims to ensure that all pupils:

- become ***fluent*** in the fundamentals of mathematics…
- ***reason mathematically***…
- can ***solve problems***…"

DfE (2013, p. 2, original emphasis)

**3** $4x + 3 = 2x + 5$

**4** $2x - 3 = x - 1$

**5** $2x + 1 = 3x - 2$

**6** $5x - 3 = 2x + 12$

**7** $4x + 9 = 8x - 31$

**8** $2x + 40 = 12x - 110$

**9** $3x + 4 = 5x - 8$

**10** $2x - 8 = 3x - 16$

**11** $x + 1 = 5x + 9$

$5x = 2x + 12$

**19** $3x + 9 = x - 5$

**20** $6x - 4 = x + 16$

**21** $x - 7 = 7x - 25$

**22** $x + 5 = 4x - 4$

**23** $6x + 5 = 3x - 7$

**24** $x + 1 = 7x - 17$

**25** $3x - 4 = 5x + 6$

**26** $8x + 3 = 6x + 15$

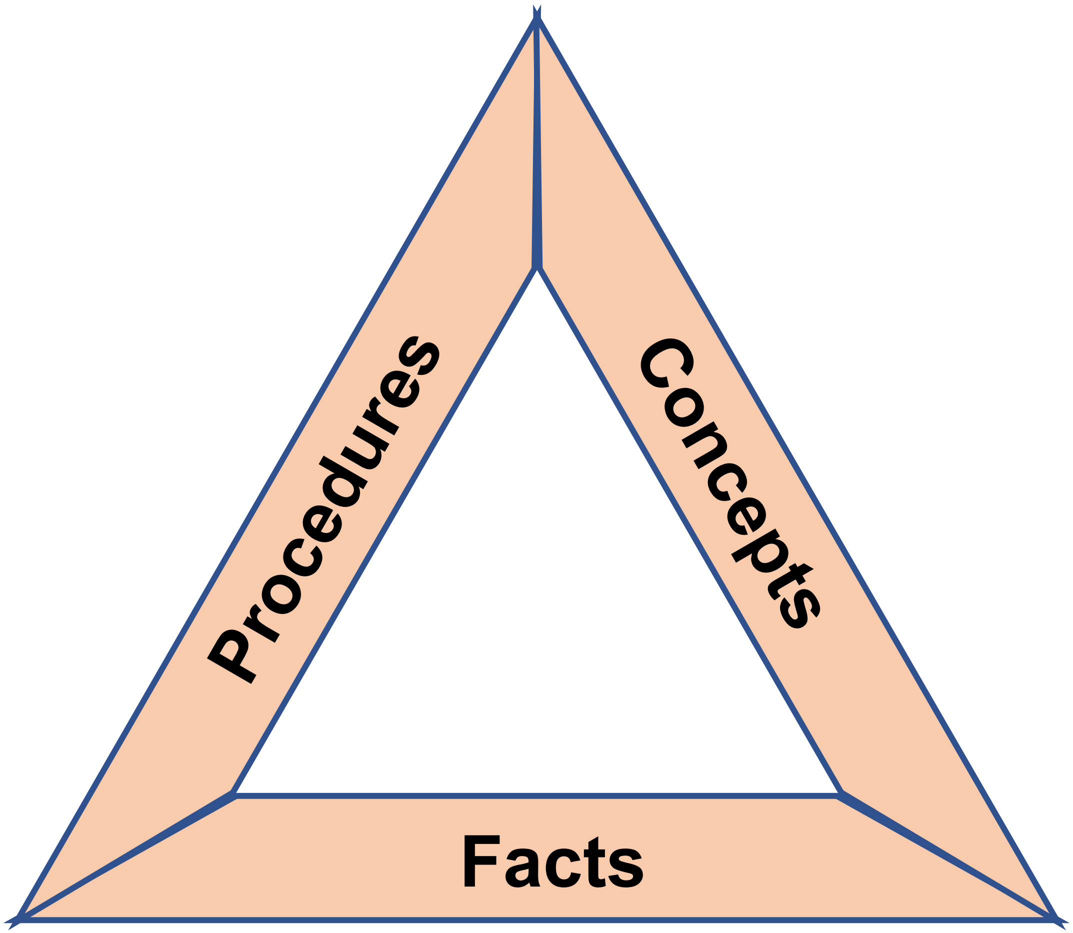
**27** $x = 20 - x$

**28** $3x - 1 = x + 7$

# Procedural Fluency

Does procedural fluency matter?

Yes.


Procedures
Concepts
Facts

# Procedural Fluency

"It is a profoundly erroneous truism … that we should cultivate the habit of thinking what we are doing. The precise opposite is the case. ~~Civilization~~ **Mathematics** advances by extending the number of important operations which we can perform without thinking about them."

(Whitehead, 1911, 58-61).

# Internalising procedures

Give opportunities for learners to develop their fluency in important mathematical procedures while something “a bit more interesting” is going on.

**Subordinating the skill**

“practice can take place without the need for what is to be practised to become the focus of attention” (p. 34)

Hewitt (1996)

# Procedural Fluency

Is it only possible to achieve it by subjecting pupils to dull, repetitive exercises?

“the notion that if a child repeats a meaningless statement or process enough times it will become meaningful is as absurd as the notion that if a parrot imitates human speech long enough it will know what it is talking about”

(Holt, 1990, p. 193)

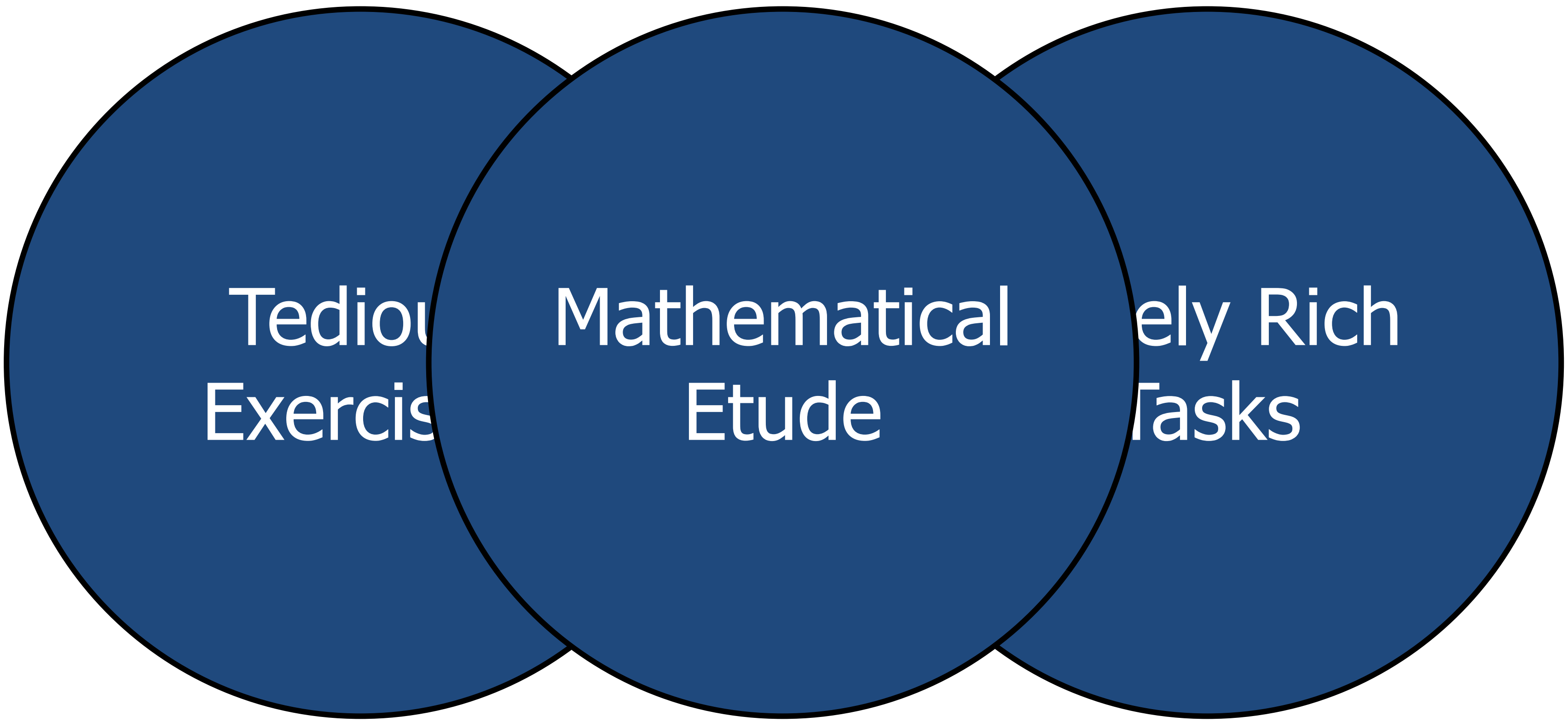
# The problem with “rich tasks”

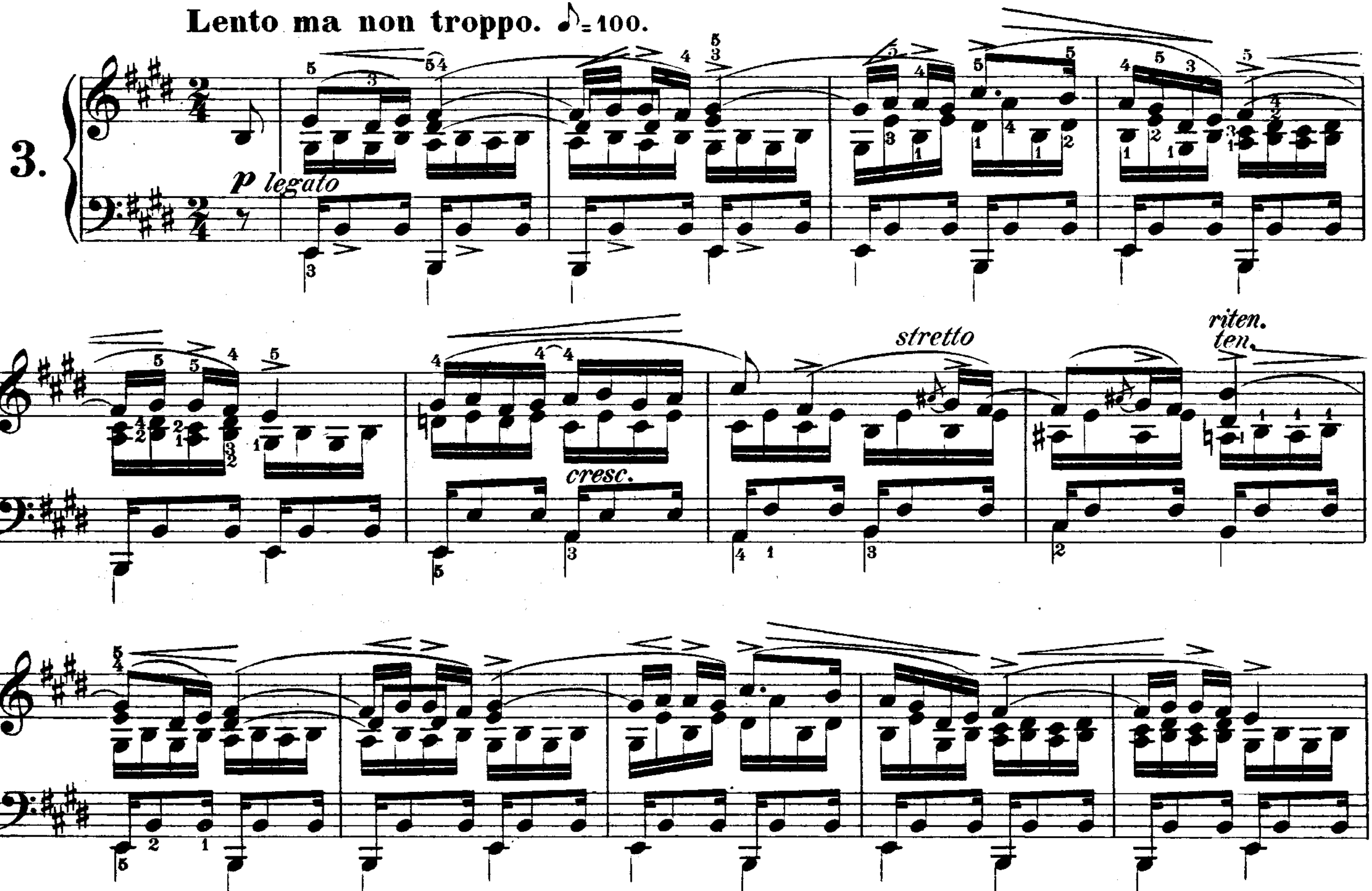
Rich
Open-ended
Exploratory
Investigative
Problem-solving

Let the fluency develop incidentally while students work on more interesting problems?

- Students take multiple approaches and learn different things
- Possibility of avoiding less comfortable areas (play to strengths, use what they know)
- Lack of concentrated focus on any one particular technique

# Mathematical Etudes




3.
Lento ma non troppo. ♪= 100.
p legato
cresc.
stretto
riten.
ten.

# Musical Étude

“originally a study or technical exercise, later a complete and musically intelligible composition exploring a particular technical problem in an esthetically satisfying manner”

*Encyclopaedia Britannica*

www.mathematical.com

**www.mathematicaletudes.com**

The **Mathematical Etudes Project** aims to find creative, imaginative and thought-provoking ways to help learners of mathematics develop their fluency in important mathematical procedures.

Procedural fluency involves knowing when and how to apply a procedure and being able to perform it "accurately, efficiently, and flexibly" (NCTM, 2014, p. 1). Fluency in important mathematical procedures is a critical goal within the learning of school mathematics, as security with fundamental procedures offers pupils increased power to explore more complicated mathematics at a conceptual level (Foster, 2013, 2014, 2015; Gardiner, 2014; NCTM, 2014). The new national curriculum for mathematics in England emphasises procedural fluency as the first stated aim (DfE, 2013).

But it is often assumed that the only way to get good at standard procedures is to drill and practise them *ad nauseum* using dry, uninspiring exercises.

The **Mathematical Etudes Project** aims to find practical classroom tasks which embed extensive practice of important mathematical procedures within more stimulating, rich problem-solving contexts (Foster, 2011, 2013, 2014). For more details see the papers listed below or scroll down for some example tasks.

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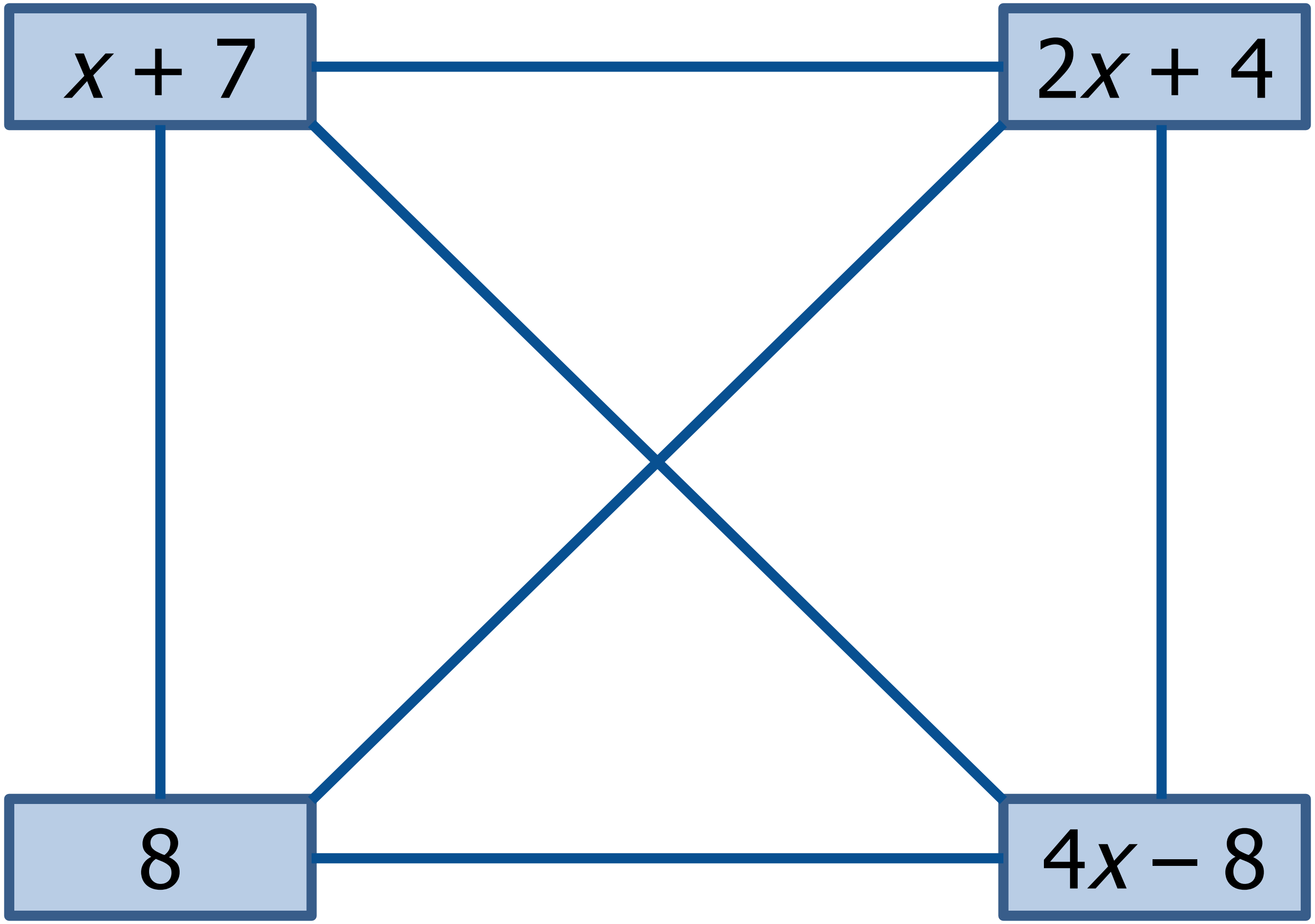



Foster, C. (2012). Connected expressions. *Mathematics in School, 41*(5), 32–33.

# Research Question

Are etudes as effective as traditional exercises at developing students' procedural fluency or not?

**Three studies:**

1. Expression polygons
2. Devising equations
3. Enlargements

# Quasi-experimental design

Two “parallel” classes, generally the same teacher across one lesson:

- Control group: complete as many short traditional exercises as possible
- Intervention group: tackle a mathematical etude on the same content

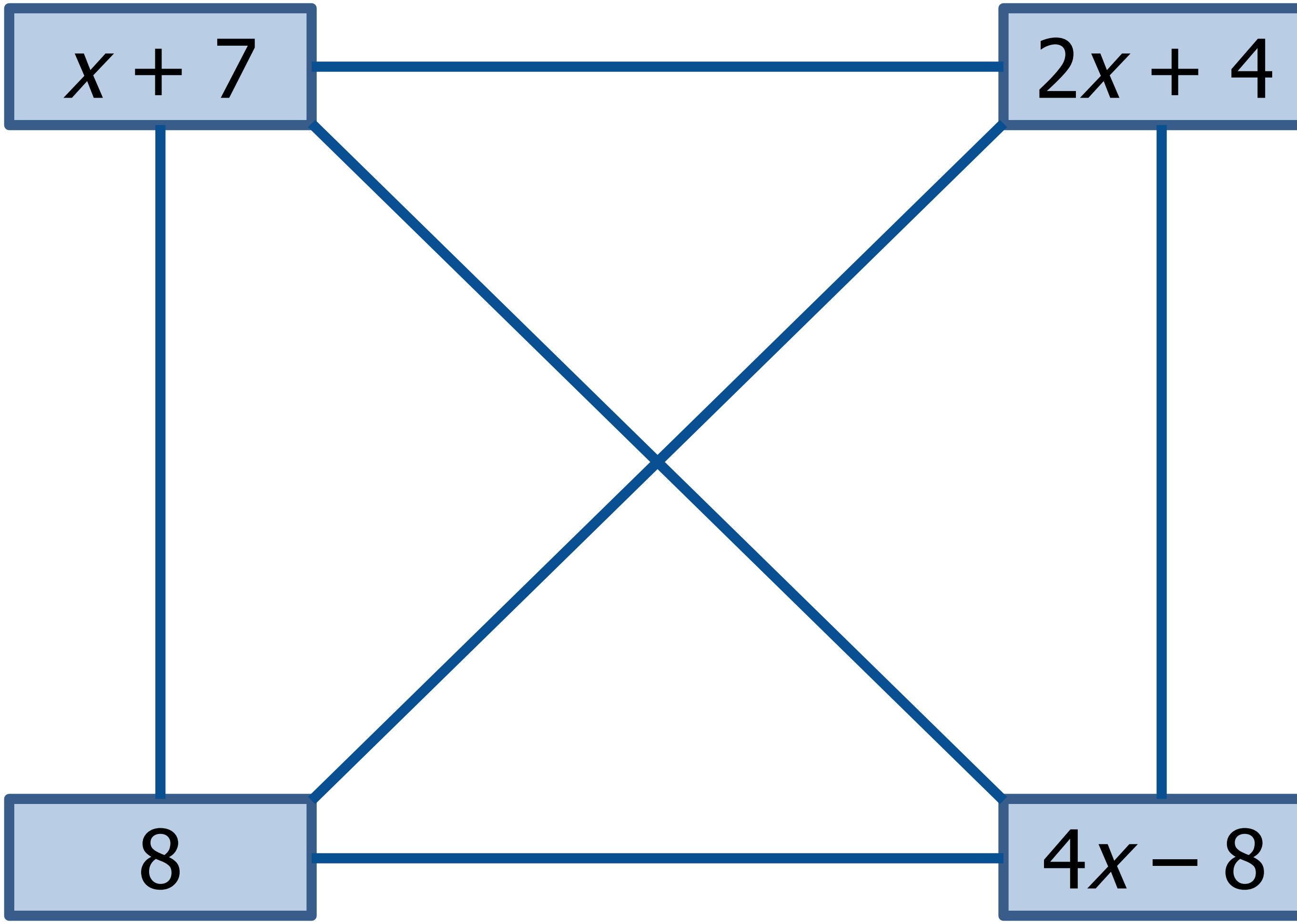
Pre- and post- tests administered at the beginning and end of the lesson.

# Participants

| | Study 1<br>Expression Polygons | Study 2<br>Devising Equations | Study 3<br>Enlargements | Total |
|---|---|---|---|---|
| *N* | 193 | 194 | 141 | 528 |
| Ages | 12-14 | 12-14 | 13-15 | 12-15 |
| Schools | 3 | 5 | 3 | 11 |

| Study | School | Location | Type | Sex | Number of students | | | School Total | Study Total |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | Y8 | Y9 | Y10 | | |
| 1 | A | London | academy | mixed | 76 | | | 76 | |
| | B | West Midlands | academy | mixed | 26 | 25 | | 51 | |
| | C | West Midlands | academy | girls | 20 | 46 | | 66 | 193 |
| 2 | D | Scotland | comprehensive | mixed | 29 | | | 29 | |
| | E | London | academy | mixed | 27 | | | 27 | |
| | F | East Midlands | academy | mixed | 86 | | | 86 | |
| | G | East Midlands | academy | mixed | | 18 | | 18 | |
| | H | Kent | academy | mixed | | 34 | | 34 | 194 |
| 3 | I | West Midlands | academy | mixed | | 52 | | 52 | |
| | J | Oxfordshire | academy | mixed | | | 47 | 47 | |
| | K | West Midlands | comprehensive | mixed | | | 42 | 42 | 141 |
| | | | | | 264 | 175 | 89 | 528 | 528 |

# Etude 1
# Expression polygons




Foster, C. (2012). Connected expressions. *Mathematics in School, 41*(5), 32–33.

# Equations BEFORE Test

Solve these four equations.

Show your method for each one.

| | |
|---|---|
| $2x + 4 = 3x + 1$ | $4x + 7 = 2x - 3$ |
| $5x - 4 = 3x + 6$ | $x - 8 = 5x - 20$ |

## Solving Equations

Solve these equations.

Show your method for each one.

**1** $2x + 4 = 3x + 1$

**2** $3x + 5 = 4x + 3$

**3** $4x + 3 = 2x + 5$

**4** $2x - 3 = x - 1$

**5** $2x + 1 = 3x - 2$

**6** $5x - 3 = 2x + 12$

**7** $4x + 9 = 8x - 31$

**8** $2x + 40 = 12x - 110$

**9** $3x + 4 = 5x - 8$

**10** $2x - 8 = 3x - 16$

**11** $x + 1 = 5x + 9$

**12** $5x = 2x + 12$

**13** $9x + 8 = 20 - 3x$

**14** $5x - 2 = x + 2$

**15** $4x + 2 = 3x + 9$

**16** $7x - 3 = 2x + 2$

**17** $3x - 5 = x + 1$

**18** $x + 6 = 2x - 5$

**19** $3x - 4 = x - 6$

**20** $3x + 9 = x - 5$

**21** $6x - 4 = x + 16$

**22** $x - 7 = 7x - 25$

**23** $x + 5 = 4x - 4$

**24** $6x + 5 = 3x - 7$

**25** $x + 1 = 7x - 17$

**26** $3x - 4 = 5x + 6$

**27** $8x + 3 = 6x + 15$

**28** $x = 20 - x$

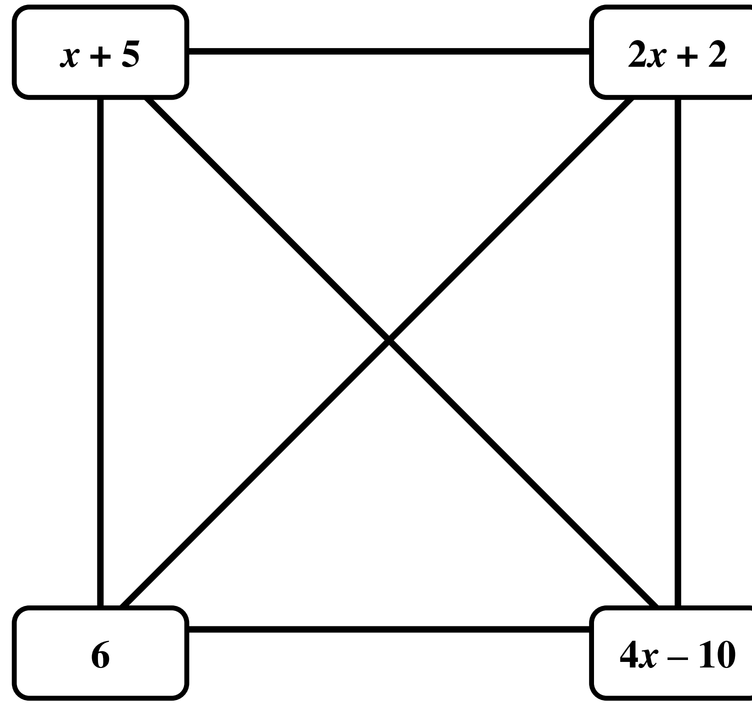
**29** $3x - 1 = x + 7$

**30** $x - 6 = 9 - 2x$

## Expression Polygons

In the diagram below, every line creates an equation.

So, for example, the line at the top gives the equation $x + 5 = 2x + 2$.




**1.** Write down and solve the six equations in this diagram.

**2.** What do you notice about your six solutions?

**3.** Now make up another diagram like this containing different expressions. Try to make the solutions to your *expression polygon* a “nice” set of numbers.

**4.** Make up some more *expression polygons* like this and see if other people can solve them.

# Instructions to the teacher

“Please allow the two classes the same amount of time to work on these sheets – however much time you have available and feel is appropriate; ideally at least a whole lesson and perhaps more. Help both classes as you would normally, using your professional judgment as to what is appropriate, so that they benefit from the time that they spend on these sheets.”

# Equations AFTER Test

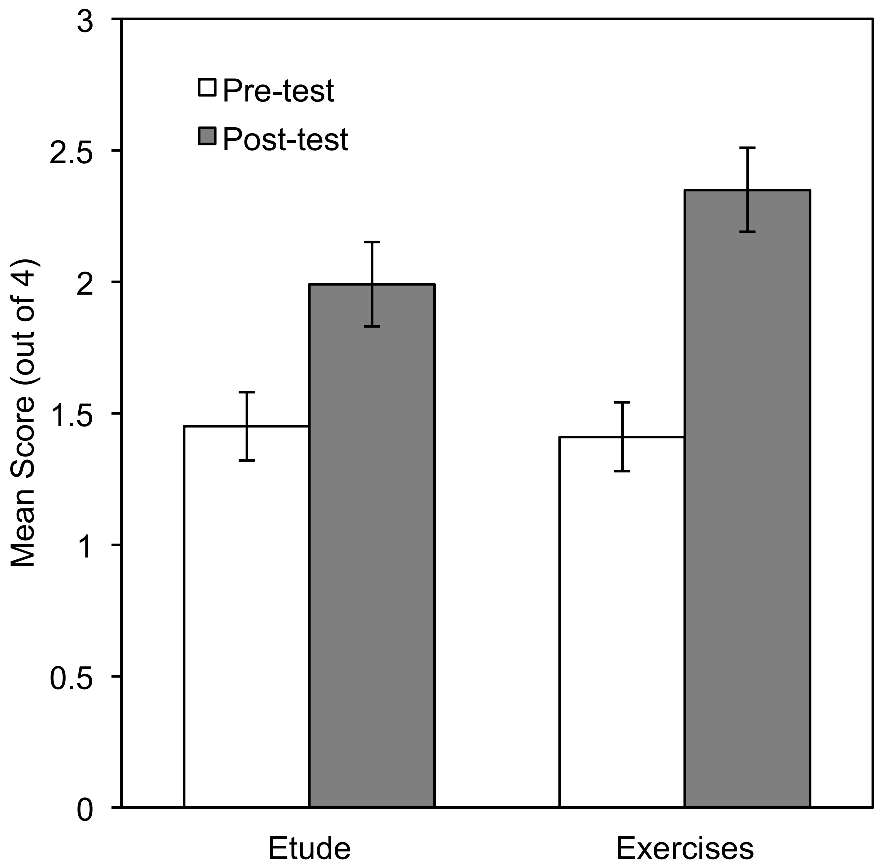
Solve these four equations.

Show your method for each one.

| | |
|---|---|
| $2x + 5 = 3x + 2$ | $4x + 5 = 2x - 3$ |
| $5x - 2 = 3x + 8$ | $x - 5 = 4x - 20$ |

Please write down below what you think about the work you have done on solving equations.

# Study 1



3
2.5
2
1.5
1
0.5
0
Mean Score (out of 4)
Pre-test
Post-test
Etude
Exercises

# Etude 2
# Devising Equations

# Equations Task

**1.** Make up an equation by choosing numbers to go in the empty boxes.

$$\square x + \square = \square x + \square$$

For example, if you chose the numbers **5**, **4**, **2** and **10**, you would get the equation $5x + 4 = 2x + 10$.

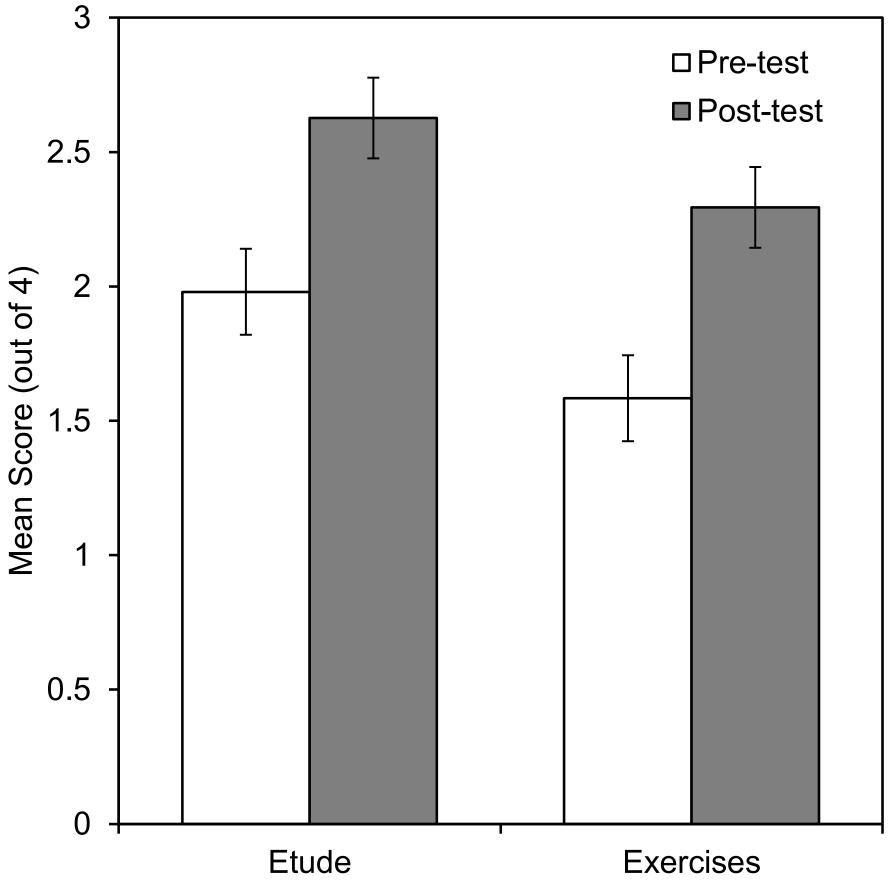
**2.** Solve your equation.

For example, when you solve the equation $5x + 4 = 2x + 10$ you get $x = 2$.

**3.** Does your equation have a whole-number answer like this one?

**4.** Choose another set of four numbers to make another equation.
Try to make as many equations as you can that have whole-number answers.

# Study 2



Mean Score (out of 4)
3
2.5
2
1.5
1
0.5
0
Pre-test
Post-test
Etude
Exercises

# Etude 3
# Enlargements

# Enlargement drawings

- "Sir, it's gone off the edge of the paper!"
- Avoid the problem with pre-prepared sheets

**or**

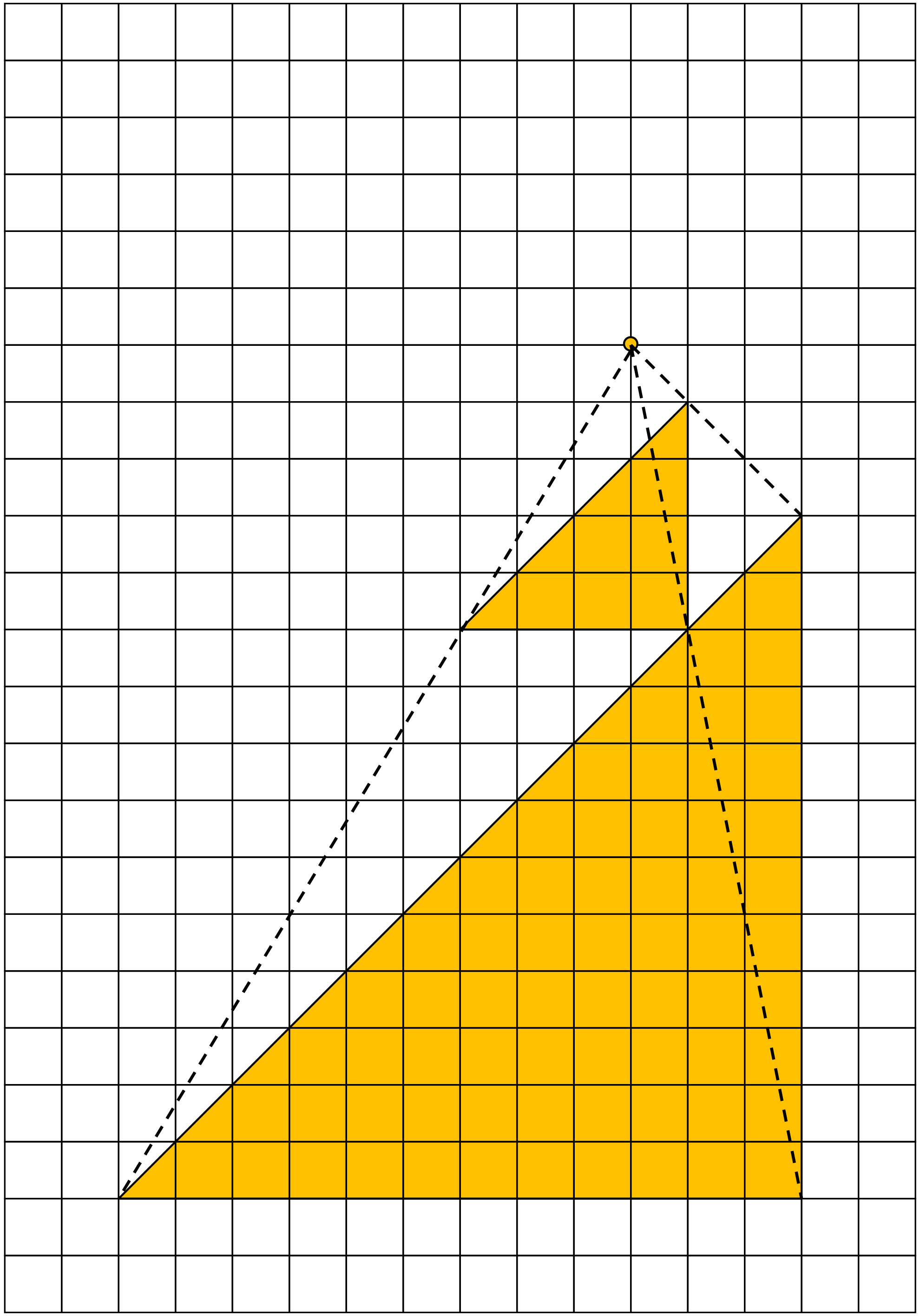
- Address the problem by making it the point of the task

# Enlargement drawings

Given an A4 piece of paper and a given shape and a given scale factor of enlargement, where can the centre of enlargement be so that all of the shape stays on the paper?

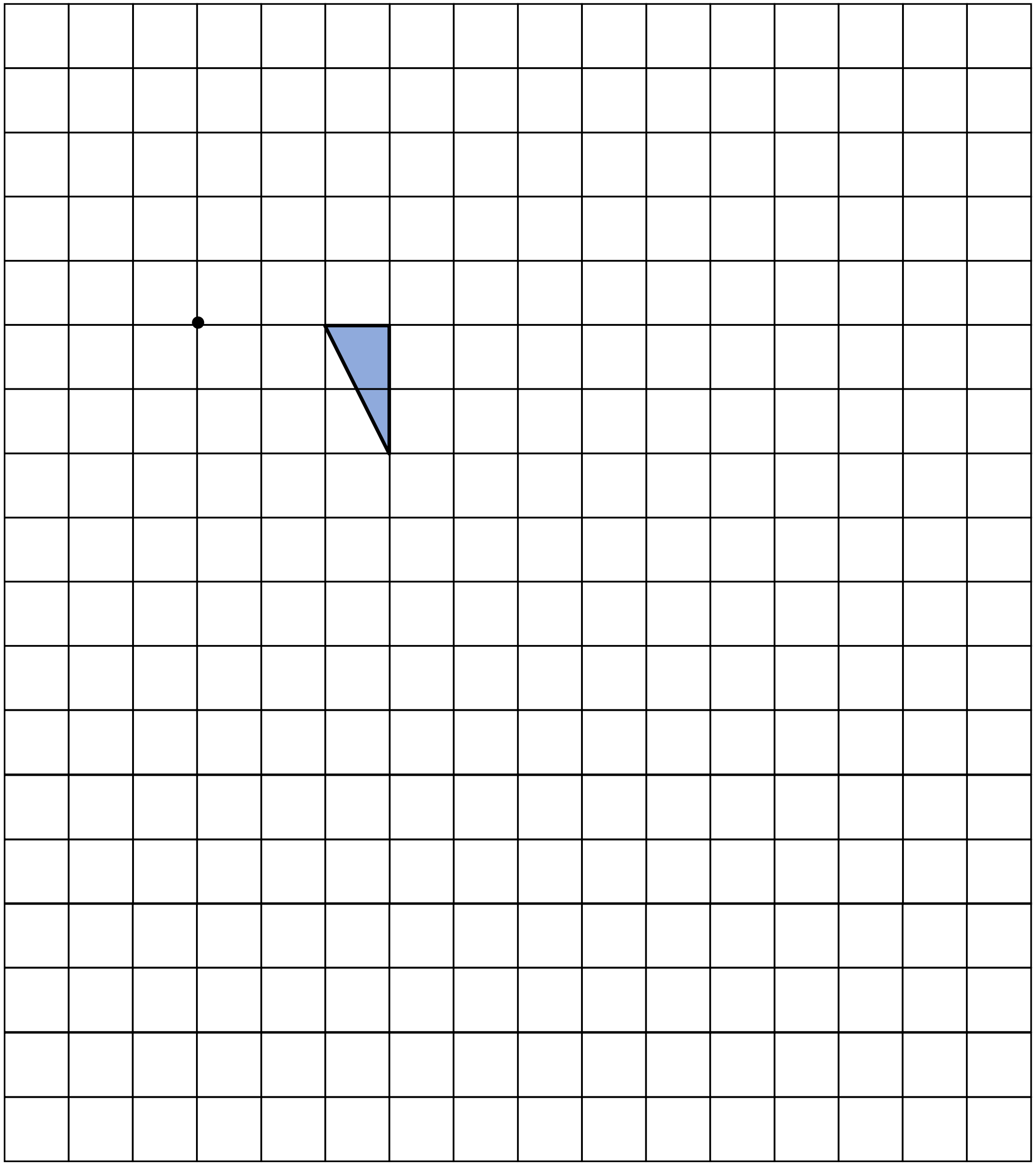
What is the locus of possible centres of enlargement for the triangle on the sheet if the scale factor is 3?

Foster, C. (2012). Working without a safety net. *The Australian Mathematics Teacher, 68*(2), 25–29.
Foster, C. (2013). Staying on the page. *Teach Secondary*, *3*(1), 57-59.



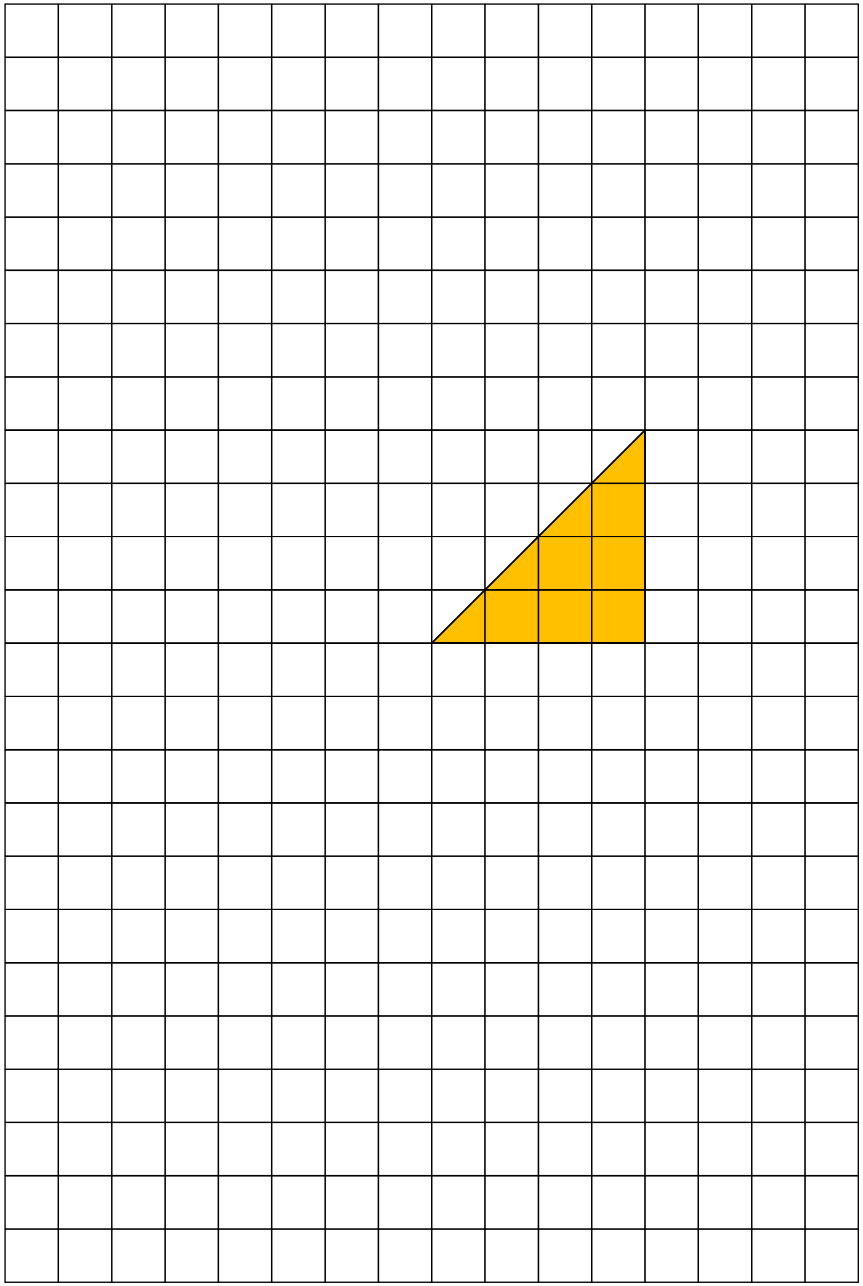
# Enlargement BEFORE Test

Enlarge the triangle below with a scale factor of 4 about the centre of enlargement marked with a dot.



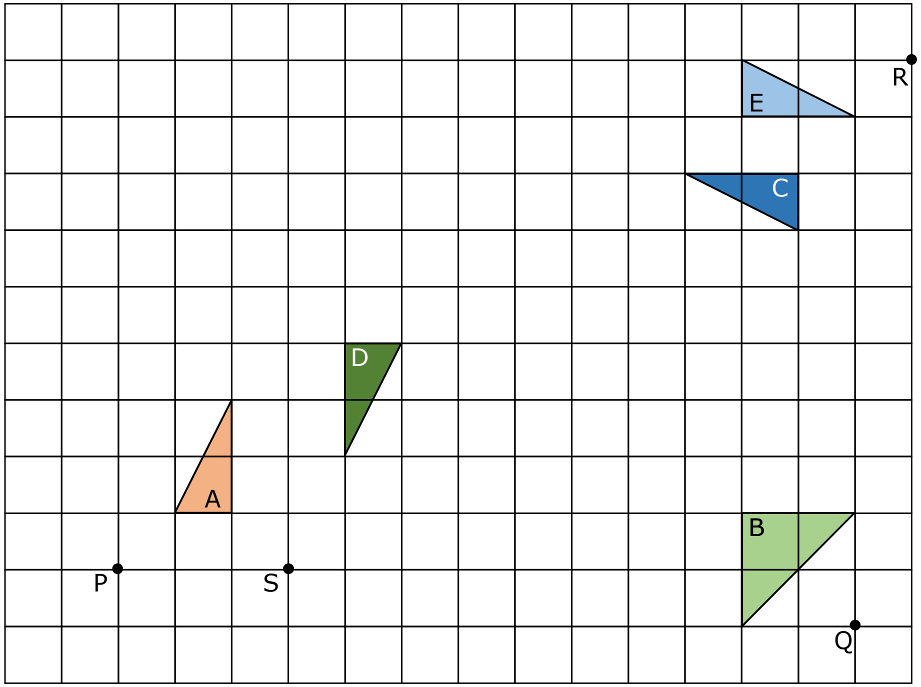
# Enlargement Task

For a scale factor 3 enlargement of this triangle, where can the centre of enlargement be so that all of the enlarged shape is on the grid?



# Enlargement Exercises

Here are five shapes (A, B, C, D and E) and four points (P, Q, R and S).




**1.** Enlarge shape A by a scale factor of 3 about centre of enlargement P.
Label your shape F.

**2.** Enlarge shape B by a scale factor of 2 about centre of enlargement Q.
Label your shape G.

**3.** Enlarge shape C by a scale factor of 3 about centre of enlargement R.
Label your shape H.

**4.** Enlarge shape A by a scale factor of 2 about centre of enlargement S.
Label your shape I.

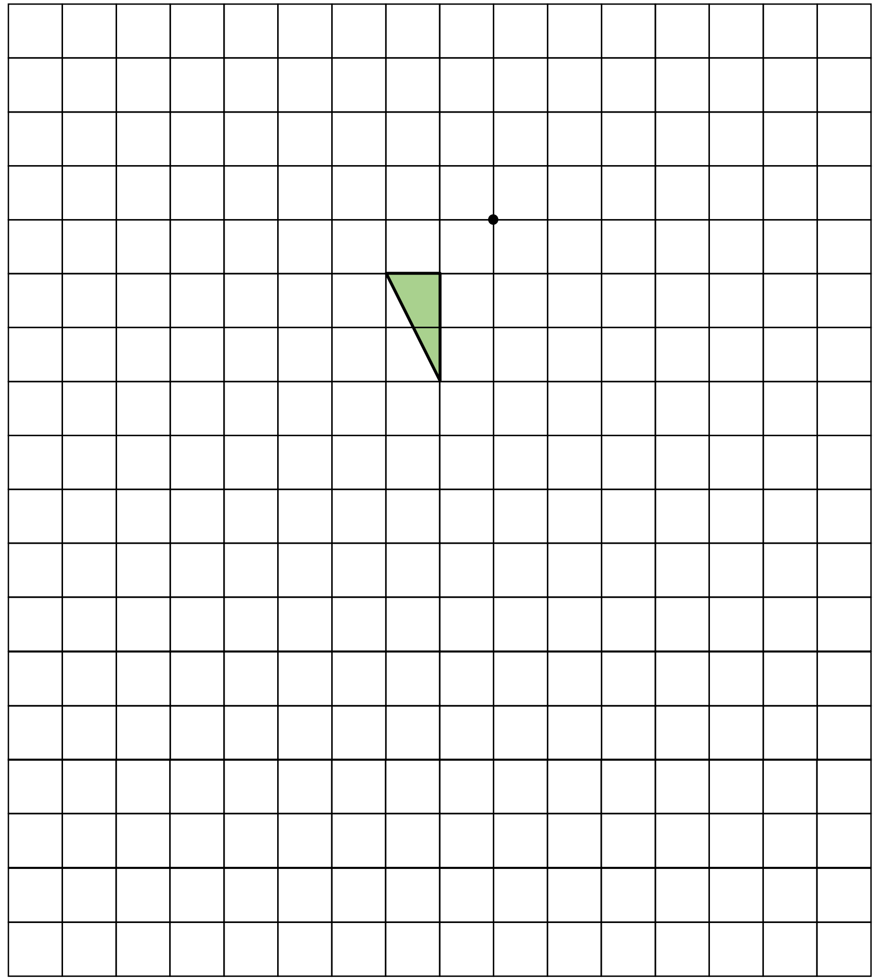
**5.** Enlarge shape D by a scale factor of 2 about centre of enlargement S.
Label your shape J.

**6.** Enlarge shape E by a scale factor of 5 about centre of enlargement R.
Label your shape K.

**7.** Enlarge shape D by a scale factor of 2 about centre of enlargement P.
Label your shape L.

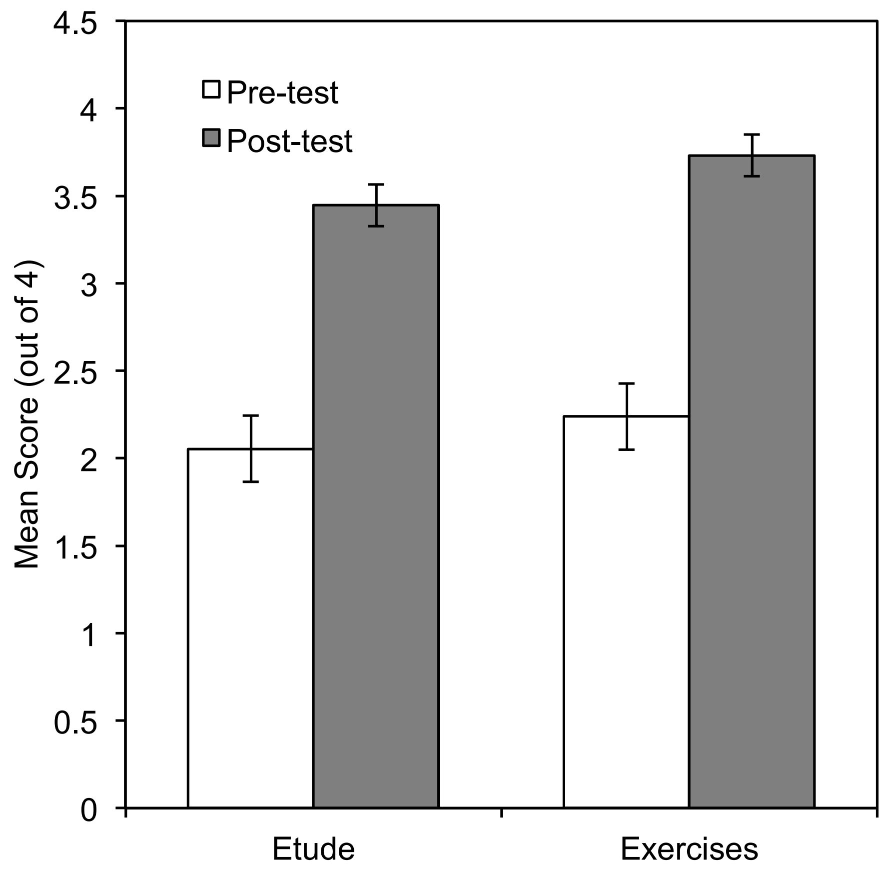
# Enlargement AFTER Test

Enlarge the triangle below with a scale factor of 4 about the centre of enlargement marked with a dot.



Please write down below what you think about the work you have done on enlargements.

# Study 3


4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
Mean Score (out of 4)
Pre-test
Post-test
Etude
Exercises

# Results

| | $M_{pre}$ | $M_{post}$ | $M_{gain}$ | $N$ | $SD_{pre}$ | $SD_{post}$ | $SD_{gain}$ |
|---|---|---|---|---|---|---|---|
| Etude | 1.45 | 1.99 | 0.542 | 107 | 1.38 | 1.33 | 1.34 |
| Exercises | 1.41 | 2.35 | 0.942 | 86 | 1.58 | 1.38 | 1.46 |

| | $M_{pre}$ | $M_{post}$ | $M_{gain}$ | $N$ | $SD_{pre}$ | $SD_{post}$ | $SD_{gain}$ |
|---|---|---|---|---|---|---|---|
| Etude | 1.98 | 2.63 | 0.647 | 75 | 1.55 | 1.48 | 1.21 |
| Exercises | 1.58 | 2.29 | 0.710 | 119 | 1.54 | 1.48 | 1.27 |

| | $M_{pre}$ | $M_{post}$ | $M_{gain}$ | $N$ | $SD_{pre}$ | $SD_{post}$ | $SD_{gain}$ |
|---|---|---|---|---|---|---|---|
| Etude | 2.05 | 3.45 | 1.39 | 74 | 1.63 | 1.24 | 1.67 |
| Exercises | 2.24 | 3.73 | 1.49 | 67 | 1.64 | 0.90 | 1.58 |

# Bayesian Statistics

- Problem with null-hypothesis significance testing:
  *either*
  - reject the null hypothesis,

  *or*
  - fail to reject the null hypothesis

  p-value is the probability of data at least as extreme as this data, *given that the null hypothesis is true*

- Bayesian statistics:
  - probability that the null hypothesis is true
  - evidence *for/against* the null hypothesis

# Bayes Factors

Bayes factors quantify how well one hypothesis predicts the data, relative to another hypothesis.

For example, a $BF_{01}$ of 3 means that the data is 3 times as probable if $H_0$ is true as it is if $H_1$ is true:

$$BF_{01} = \frac{p(D|H_0)}{p(D|H_1)}$$

where *D* is the data obtained.

# Bayesian t tests

- Cauchy prior width of .707
- *BayesFactor* package in R
- meta.ttestBF Bayesian meta-analysis function

| *Study* | *Bayes factor* |
|---|---|
| 1 | 1.03 |
| 2 | 5.92 |
| 3 | 5.20 |
| *Combined* | 5.83 |

# Summary

Are etudes as effective as traditional exercises at developing students' procedural fluency or not?

Yes. Some evidence that they are equally effective.

- 528 Year 7-9 mathematics students from 11 secondary schools
- Quasi-experimental design, trialling 3 etudes, each against a corresponding traditional exercise
- Statistical analysis gave an overall Bayes factor of 5.83, constituting "substantial" evidence in favour of the null hypothesis of no difference

# Conclusion

Even if all you care about is that students develop lots of procedural fluency ...

... you might as well use etudes!

And they are likely to have other (harder-to-measure) benefits too:

- Creative investigative inquiry
- Surprise and motivation
- Communication
- ...

# Principles for creating mathematical études

- Inviting learners to modify questions, perhaps in order to achieve some particular result
- Practising a procedure while searching for something a bit more "profound"
- Starting with the answer and working back to possible questions

Prestage & Perks (2001), Watson & Mason (2005), Foster (2011, 2013)

# Further Work

- Try to specify design principles
- Try to quantify harder-to-measure possible *benefits* of etudes
- Delayed post-tests
- Examine in a finer level of detail what happens differently with etudes
- Other ages and topics

# Acknowledgements

Thank you to the pupils and teachers who participated in the etudes studies.

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